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The Computational Complexity and Approximability of a Series of Geometric Covering Problems

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Abstract—We consider a series of geometric problems of covering finite subsets of finite-dimensional numerical spaces by minimal families of hyperplanes. We prove that the problems are hard and Max-SNP-hard.

Keywords: NP-complete problem, polynomial-time reduction, Max-SNP-hard problem, L-reduction, polynomial-time approximation scheme (PTAS).

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INTRODUCTION

Statements of geometric problems of minimal covering and related problems of combinatorial optimization arise in different fields of operations research: in the theory of optimal facility location, in cluster analysis, in machine learning theory [1–3]. Mathematically, the family of such problems can be divided into two classes.

The first class consists of problems that are variations of the known abstract problem of covering a set (Set Cover). The main feature characterizing these statements is the finiteness of the original family of subsets in which it is required to find an optimal (in the sense of minimal cardinality, minimal total weight, and so on) subfamily covering a given target set. A large number of papers (see review in [4]) are devoted to the investigation of this class of problems. It seems that, among these works, the most important are classical papers [5,6], containing the proof of the intractability of the Set Cover problem and the description of two basic approaches to constructing polynomial time approximation algorithms for its solution, as well as paper [7] devoted to the substantiation of the order optimality of the algorithms by D. Johnson and L. Lovász (under the assumption $P \neq NP$).

The second class comprises covering problems in which the additional constraint of the finiteness of the family of covering subsets is absent. As a rule, in such problems, this family is given implicitly by specifying a general geometric property inherent to its elements. For example, it is required to find a minimal covering of a specified set by straight line segments, circles of a given radius, etc.

In the present paper, we study a series of problems of covering a finite subset of a finite-dimensional vector space of fixed dimension by hyperplanes. Seemingly, a similar planar problem

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was first considered in Megiddo and Tamir's paper [8], where its intractability in the strong sense was proved. Below, this result is extended to the case of arbitrary fixed dimension $k > 1$. We also consider the question of the effective approximability of the investigated series of problems. In particular, it is shown that all the problems are Max-SNP-hard, and, consequently, a polynomial time approximation scheme cannot be constructed for them under the condition $P \neq NP$.

1. DEFINITIONS AND PROBLEM STATEMENTS

This section contains statements of problems of combinatorial optimization investigated in the present paper, main definitions, and a review of some known results necessary for further reasoning.

Let a set X and a nonempty family of its subsets

$$\mathcal{C} = \{C_\alpha \mid \alpha \in \Lambda\}$$

be given. As usual, we call the family $\mathcal{C}$ a *covering* of a subset $A \subset X$ if, for any $a \in A$, there exists an element $C_\alpha \in \mathcal{C}$ such that $a \in C_\alpha$.

Everywhere below, we assume that X is a finite-dimensional numerical space and elements C_α are its *hyperplanes*, i.e., proper affine subspaces of maximal dimension. We are interested in minimal coverings (by hyperplanes) of finite subsets of the space X.

Problem 1: "Covering a finite subset of the plane by straight lines" (2PC). A finite subset of the plane $P = \{p_1, \ldots, p_n\}$ with integer coordinates and a number $B \in \mathbb{N}$ are given. Does there exist a covering of the set P by a set of straight lines with cardinality not exceeding B?

Evidently, if the set P is in general position, i.e., no three points from this set belong to the same straight line, then the problem 2PC has a trivial solution (positive if $B \geq \lceil |P|/2 \rceil$ and negative otherwise). Moreover, this solution can be found in a time upper bounded by a polynomial of the instance length of the problem. Nevertheless, for the general case, the following result is known.

Theorem 1 [8]**.** *The problem* 2PC *is NP-complete in the strong sense.*

In our paper, the statement of the covering problem is naturally extended to the case of spaces of larger dimension.

Problem 2: "Covering a finite subset of a k-dimensional space by hyperplanes" (kPC). For some fixed $k > 1$, a finite subset $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{Z}^k$ and a number $B \in \mathbb{N}$ are given. Does there exist a covering of the set P by a set of hyperplanes with cardinality not exceeding B?

In Section 2, the result of Theorem 1 is generalized to the case of the problem of covering by hyperplanes (kPC) in a space of arbitrary fixed dimension $k > 1$.

In this paper, along with the recognition problem, an optimization version of the problem of covering by hyperplanes is considered.

Problem 3: "The problem of minimal covering of a finite subset of a k-dimensional space by hyperplanes" (Min-kPC). Let a finite set

$$P = \{p_1, \ldots, p_n\} \subset \mathbb{Z}^k$$

be given. It is required to find a minimal decomposition $J_1, \ldots J_L$ of the set $\mathbb{N}_n = \{1, \ldots, n\}$ such that a hyperplane H_i possessing the property

$$\{p_j \in P \colon j \in J_i\} \subset H_i$$

can be assigned to each $i \in \mathbb{N}_L$.

An important direction of investigating NP-hard problems of combinatorial optimization is connected with studying the possibility of constructing a polynomial time approximation scheme (PTAS) for a specific problem. The notion of *L-reduction*, introduced for the first time by Papadimitriou and Yannakakis [9], is analogous to the notion of polynomial reducibility in the theory of NP-complete problems and allows us to extend the results on the possibility (or impossibility) of constructing such schemes to new types of problems.

Let us give the definition of L-reduction following monograph [10].

Definition 1. Let sets $\mathfrak{I}$ and S, a point-set mapping $F\colon \mathfrak{I} \to 2^S$, and a target function $c\colon \bigcup_{I\in\mathfrak{I}} F(I) \to \mathbb{R}_+$ be given. The ordered quadruple $A = (\mathfrak{I}, S, F, c)$ is called a (*general*) *problem of combinatorial minimization* if, to each value $I \in \mathfrak{I}$, we assign the optimization problem

$$\min\{c(s)\colon s \in F(I)\}, \tag{1.1}$$

which is called an *instance* of the problem A.

If any ambiguous interpretation is absent, the element $I \in \mathfrak{I}$ itself is called an instance of the problem A. The optimal value of problem (1.1) is denoted by $OPT(I)$.

Definition 2. Let A and B be two problems of combinatorial minimization. There exists an *L-reduction* of the problem A to the problem B if there exist a pair of functions R and S calculated by an algorithm with logarithmic memory size and positive constants α and β such that

(1) if I is an instance of the problem A with optimal value $OPT(I)$, then $R(I)$ is an instance of the problem B with optimal value satisfying the inequality

$$OPT(R(I)) \le \alpha\, OPT(I);$$

(2) if z is an admissible solution of problem $R(I)$, then $S(z)$ is an admissible solution of the problem I such that

$$c_A(S(z)) - OPT(I) \le \beta\,(c_B(z) - OPT(R(I))),$$

where c_A and c_B are target functions of the problems A and B, respectively.

Let algorithms with logarithmic memory size be briefly called LSPACE-algorithms. The key property of L-reduction consists in the fact that it preserves the approximability property.

Assertion 1. *If there exist an L-reduction from A to B and a PTAS for B, then the problem A possesses a PTAS as well.*

In [9], the notion of complexity class Max-SNP of combinatorial optimization problems was introduced for the first time. The construction of this class is based on L-reduction. Note an important property of problems that are complete with respect to this class.

Assertion 2. *If $P \ne NP$ then no* Max-SNP*-complete problem can possess a PTAS.*

As shown in [10], the problem Max-3SAT, as well as its modification[3] Max-3SAT(t) for arbitrary $t > 2$, is Max-SNP-complete. In [11], a scheme of polynomial reduction of the problem Max-3SAT(t) to the problem Min-2PC was proposed. This reduction preserves the approximation accuracy; thus, it is shown that the latter problem is Max-SNP-hard and, consequently, has no PTAS (under the assumption $P \ne NP$).

Let φ be a 3-CNF determining the condition of some special problem Max-3SAT(t). Denote by m the number of clauses in φ and by $OPT(\varphi)$ the optimal value of the problem (the maximal

[3] The problem "3-SAT" under the additional condition that each variable can enter the Boolean formula no more than t times.

number of simultaneously solvable clauses). By analogy, let us introduce the notation $OPT(PC)$ for the optimal value of the problem Min-2PC (the cardinality of a minimal covering).

Theorem 2 [11]**.** *There exists a scheme of polynomial reduction of the problem* Max-3SAT(t) *to the problem* Min-2PC*, transforming the Boolean formula* φ *to an instance of the problem* Min-2PC *in such a way that*

- *if* $OPT(\varphi) = m$*, then* $OPT(PC) = nt$*,*
- *if* $OPT(\varphi) = m' < (1-\varepsilon)m$*, then* $OPT(PC) > nt + \lceil \varepsilon n/6 \rceil$*,*

where $\varphi = E_1 \wedge \ldots \wedge E_m$ *is a Boolean formula of* n *variables,* $\varepsilon > 0$*.*

In Section 3, an L-reduction of the problem Min-$(k-1)$PC to the problem Min-kPC for arbitrary natural $k > 2$ and, consequently, the Max-SNP-intractability of the whole family of problems are substantiated.

2. INTRACTABILITY

This section is devoted to the proof of the intractability of the problem kPC. As mentioned above, the problem 2PC is NP-complete in the strong sense. The polynomial reducibility of the problem $(k-1)$PC to the problem kPC (for arbitrary $k > 2$) substantiated in this section makes it possible to extend this property recursively to the case of arbitrary natural $k > 1$. The section consists of two subsections. In the first subsection, the reducibility of the problem 2PC to the problem 3PC and some related results are discussed. The second subsection is devoted to transferring a part of the results (in particular, the polynomial reducibility) to the case of arbitrary $k > 2$. The planar case is considered separately, since not all the results obtained for it follow immediately from the results that are valid for arbitrary dimension. Therefore, in the authors' opinion, these results are of independent interest.

2.1. Planar case. Let us show that the problem 2PC can be reduced to the problem 3PC in polynomial time. Let an instance of the problem 2PC be given by a finite subset $P = \{p_1, \ldots, p_n\}$ of the plane xOy and a positive integer B. Without loss of generality, one can assume that $P \subseteq \mathbb{N}_M^2$, where $\mathbb{N}_M = \{1, \ldots, M\}$ and $M > 1$. Let us introduce the notation $K = 2(M-1)^2$ and, to each point $p_i \in P$, assign the pair of points in the three-dimensional space with the coordinates

$$\bar{p}_{2i-1} = [p_i, -(K+2)^{i-1}] \qquad \text{and} \qquad \bar{p}_{2i} = [p_i, (K+2)^{i-1}]. \tag{2.1}$$

The points $\bar{p}_{2i-1}$ and $\bar{p}_{2i}$ are said to be *generated* by the common point p_i. Here and below, we use the notation $[x_1, x_2, \ldots, x_n]$ to denote a point (a vector) with given coordinates. For a vector $p = [x, y]$ and a number z, the notation $[p, z]$ is used to denote briefly the vector $[x, y, z]$. Thus, we construct the subset $\bar{P} \subseteq \mathbb{Z}^3$, which specifies, together with the number B, an instance of the problem 3PC.

To an arbitrary covering of the set P by straight lines L, one can naturally assign a covering of the set $\bar{P}$ by planes. For this, it is sufficient to consider the planes passing through the straight lines of the set L orthogonally to the original plane xOy.

On the other hand, let us show that the existence of a covering of the set $\bar{P}$ by planes implies the existence of a covering of the set P by straight lines with cardinality not exceeding the cardinality of the original covering. To do this, we prove several preliminary statements.

Lemma 1. *No three points from the set* $\bar{P}$ *belong to the same straight line.*

Proof. Consider an arbitrary subset

$$\{\bar{p}_1, \bar{p}_2, \bar{p}_3\} \subset \bar{P},$$

in which $\bar{p}_i = [x_i, y_i, z_i]$. As above, denote by $p_i = [x_i, y_i]$ the projection of the point $\bar{p}_i$ on the plane xOy. It is required to show that $\dim \operatorname{aff}(\{\bar{p}_1, \bar{p}_2, \bar{p}_3\}) > 1$. Evidently, the validity of the statement of the lemma in the case $|\{p_1, p_2, p_3\}| < 3$ follows from the choice of the coordinates z_i; hence, everywhere below, we assume that $|\{p_1, p_2, p_3\}| = 3$. In addition, without loss of generality, we assume that the inequalities $|z_2| \geq (K+2)|z_1|$ and $|z_3| \geq (K+2)|z_2|$ are valid.

Let, by contradiction, the points $\bar{p}_1$, $\bar{p}_2$, and $\bar{p}_3$ belong to the same straight line. Then, under our assumptions, there exists a number $t \neq 0$ such that

$$x_3 - x_1 = t(x_2 - x_1), \qquad y_3 - y_1 = t(y_2 - y_1), \qquad z_3 - z_1 = t(z_2 - z_1).$$

Assuming that $x_2 \neq x_1$ (otherwise, we use a similar estimate for y_1, y_2, and y_3), due to the fact that the coordinates are integer, we have, on the one hand,

$$M - 1 \geq |x_3 - x_1| = |t|\,|x_2 - x_1| \geq |t|. \tag{2.2}$$

On the other hand,

$$(K+2)|z_2| - \frac{|z_2|}{K+2} \leq |z_3| - |z_1| \leq |z_3 - z_1| = |t|\,|z_2 - z_1| \leq |t|\,(|z_2| + |z_1|) \leq |t|\,|z_2|\Big(1 + \frac{1}{K+2}\Big);$$

i.e.,

$$\frac{1}{K+2}|z_2|\,((K+2)^2 - 1) \leq \frac{1}{K+2}|z_2|\,|t|\,((K+2) + 1),$$

which implies

$$K + 1 \leq |t|. \tag{2.3}$$

Combining relations (2.2) and (2.3), we obtain the inequality

$$M - 1 \geq K + 1 = 2(M-1)^2 + 1,$$

which is contradictory for arbitrary M. Consequently, the assumption that the points $\bar{p}_1$, $\bar{p}_2$, and $\bar{p}_3$ belong to the same straight line is not valid.

The lemma is proved.

Note that method (2.1) for constructing the set $\bar{P}$ is not a unique way to provide the validity of Lemma 1. A similar result, for example, can be obtained if the set $\bar{P}$ is given by the rule

$$\bar{p}_{2i-1} = [p_i, -K^{i-1}] \qquad \text{and} \qquad \bar{p}_{2i} = [p_i, K^{i-1}] \qquad (p_i \in P).$$

Lemma 2. *If arbitrary four points from the set $\bar{P}$ belong to the same plane, then the points generating them, which are elements of the set P, belong to the same straight line.*

Proof. Indeed, let points $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in \bar{P}$ with coordinates $[x_a, y_a, z_a]$, $[x_b, y_b, z_b]$, $[x_c, y_c, z_c]$, $[x_d, y_d, z_d]$, respectively, belong to some plane π. Consider two cases.

1. Among the points $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$, there exists a pair of points generated by the same point $p_i \in P$. Let this pair be $\bar{a}$ and $\bar{b}$: $\bar{a} = [p_i, (K+2)^{i-1}]$ and $\bar{b} = [p_i, -(K+2)^{i-1}]$. In this case, the plane π is orthogonal to the plane xOy, and, consequently, the preimages of the points $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$ belong to the same straight line, which is the intersection of the planes π and xOy.

2. There are no two points from the set $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}$ generated by the same point $p_i \in P$. Then, we can assume that the generating points $p_i, p_j, p_k, p_l \in P$ (respectively) are such that $1 \leq i < j < k < l \leq n$. Then, by construction,

$$|z_d| \geq (K+2)\,|z_c| \geq (K+2)^2|z_b| \geq (K+2)^3|z_a|.$$

Since the points $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$ belong to the same plane, the vectors $\bar{b}-\bar{a}$, $\bar{c}-\bar{a}$, and $\bar{d}-\bar{a}$ are coplanar; consequently,

$$\Delta = \begin{vmatrix} x_b - x_a & y_b - y_a & z_b - z_a \\ x_c - x_a & y_c - y_a & z_c - z_a \\ x_d - x_a & y_d - y_a & z_d - z_a \end{vmatrix} = 0.$$

Let us expand Δ along the last column:

$$\Delta = (z_d - z_a)\Delta_d - (z_c - z_a)\Delta_c + (z_b - z_a)\Delta_b = z_d\Delta_d - z_c\Delta_c + z_b\Delta_b - z_a(\Delta_d - \Delta_c + \Delta_b),$$

where

$$\Delta_d = \begin{vmatrix} x_b - x_a & y_b - y_a \\ x_c - x_a & y_c - y_a \end{vmatrix}, \qquad \Delta_c = \begin{vmatrix} x_b - x_a & y_b - y_a \\ x_d - x_a & y_d - y_a \end{vmatrix}, \qquad \Delta_b = \begin{vmatrix} x_c - x_a & y_c - y_a \\ x_d - x_a & y_d - y_a \end{vmatrix}.$$

Since the coordinates x, y of the points $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$ belong to $\mathbb{N}_M$, each of the determinants either is zero or its absolute value is no less than one. On the other hand, the obvious upper estimate

$$\max\{|\Delta_b|, |\Delta_c|, |\Delta_d|\} \le 2(M-1)^2 = K$$

is valid. Thus, for arbitrary $t \in \{b, c, d\}$, we have

$$(\Delta_t = 0) \vee (1 \le |\Delta_t| \le K).$$

Let us show that the equality $\Delta = 0$ implies $\Delta_b = \Delta_c = \Delta_d = 0$. Assume, by contradiction, that $\Delta = 0$ and $\Delta_t \ne 0$ for some $t \in \{b, c, d\}$. Three alternatives are possible:

$$\Delta_d = \Delta_c = 0, \quad \Delta_b \ne 0, \tag{2.4}$$

$$\Delta_d = 0, \quad \Delta_c \ne 0, \tag{2.5}$$

$$\Delta_d \ne 0. \tag{2.6}$$

However, the possibility of case (2.4) is refuted by the estimate

$$|\Delta| = |z_b\Delta_b - z_a\Delta_b| \ge |z_b|\,|\Delta_b| - |z_a|\,|\Delta_b| \ge |z_b| - K\,|z_a| \ge (K+2)\,|z_a| - K\,|z_a| > 0;$$

of case (2.5), by the estimate

$$|\Delta| = |-z_c\Delta_c + z_b\Delta_b - z_a(\Delta_b - \Delta_c)| \ge |z_c|\,|\Delta_c| - |z_b|\,|\Delta_b| - |z_a|\,(|\Delta_b| + |\Delta_c|)$$
$$\ge |z_c| - K\,|z_b| - 2K\,|z_a| \ge (K+2)\,|z_b| - K\,|z_b| - 2K\,|z_a| = 2\,(|z_b| - K\,|z_a|) > 0;$$

and of case (2.6), by the estimate

$$|\Delta| = |z_d\Delta_d - z_c\Delta_c + z_b\Delta_b - z_a(\Delta_b - \Delta_c + \Delta_d)| \ge |z_d| - K\,|z_c| - K\,|z_b| - 3K\,|z_a|$$
$$\ge (K+2)\,|z_c| - K\,|z_c| - K\,|z_b| - 3K\,|z_a| \ge 2\,|z_c| - K\,|z_b| - 3K\,|z_a| \ge 2\,(|z_c| - K\,|z_b| - 2K\,|z_a|) > 0.$$

Thus, the fact that the determinant Δ is zero (the coplanarity of the vectors $\bar{b}-\bar{a}$, $\bar{c}-\bar{a}$, and $\bar{d}-\bar{a}$) implies the equalities $\Delta_b = \Delta_c = \Delta_d = 0$, which, in turn, imply that the points $p_i = [x_a, y_a]$, $p_j = [x_b, y_b]$, $p_k = [x_c, y_c]$, and $p_l = [x_d, y_d]$ belong to the same straight line.

The lemma is proved.

Corollary 1. *A plane π containing arbitrary points $\bar{p}_1, \ldots, \bar{p}_4 \in \bar{P}$, $\bar{p}_i = [p_i, z_i]$, $p_i \in P$, contains also the points p_i and $[p_i, -z_i]$, $i \in \mathbb{N}_4$, and is orthogonal to the plane xOy.*

Proof. Indeed, by Lemma 2, the points $p_1, \ldots, p_4$ belong to a straight line l (in the plane xOy). The plane π' orthogonal to the plane xOy and intersecting it in the line l, evidently, contains all the points specified above:

$$p_1, \ldots, p_4, \qquad \bar{p}_1 = [p_1, z_1], \ldots, \bar{p}_4 = [p_4, z_4], \qquad [p_1, -z_1], \ldots, [p_4, -z_4].$$

On the other hand, by Lemma 1, the points $\bar{p}_1$, $\bar{p}_2$, and $\bar{p}_3$ do not belong to the same straight line; consequently, the plane containing them is uniquely determined, which implies $\pi = \pi'$.

Lemma 3. *Let C be an arbitrary covering of the set $\bar{P}$ by planes. Then, there exists a covering of the set P by straight lines with cardinality not exceeding the cardinality of C.*

Proof. Consider an arbitrary covering C of the set $\bar{P}$ by planes and divide it into two classes, C_1 and C_2. To the first class, we assign the planes that are orthogonal to xOy; to the second class, all the remaining planes. Denote by $\bar{P}_1$ the subset of $\bar{P}$ covered by planes from the class C_1 and by $\bar{P}_2 = \bar{P} \setminus \bar{P}_1$ its complement. Let us introduce the notation P_1 and P_2 for the projections onto the plane xOy of the subsets $\bar{P}_1$ and $\bar{P}_2$, respectively. Note that, by Corollary 1, the points $\bar{p}_i = [p_i, z_i]$ and $[p_i, -z_i]$ belong (or do not belong) to the subset $\bar{P}_1$ simultaneously; consequently, $|\bar{P}_i| = 2|P_i|$. By Lemma 2, an equinumerous covering L_1 of the set P_1 by straight lines corresponds to the class C_1.

Let $\bar{P}_2 \neq \varnothing$. By Lemma 2, any plane that is an element of the class C_2 contains at most three elements of the set $\bar{P}_2$; i.e., $|C_2| \geq \lceil |\bar{P}_2|/3 \rceil$. On the other hand, the subset P_2 possesses a covering L_2 containing at most

$$\left\lceil \frac{|P_2|}{2} \right\rceil = \left\lceil \frac{|\bar{P}_2|}{4} \right\rceil$$

straight lines; consequently, the cardinality of the covering $L_1 \cup L_2$ of the set P by straight lines does not exceed $|C_1| + |C_2| = |C|$.

The lemma is proved.

The following lemma completes the substantiation of the polynomial reduction of the problem 2PC to the problem 3PC.

Lemma 4. *The reduction of the problem* 2PC *to the problem* 3PC *described above can be performed in polynomial time with respect to the length of the problem* 2PC.

Proof. An instance of the problem 2PC is given by a set of points

$$P = \{p_1, \ldots, p_n\} \subseteq \mathbb{N}_M^2$$

and a number $B \in \mathbb{N}$. Consequently, the length of the problem is as follows:[4]

$$\mathrm{Len}_1 = 2n \log M + \log B \geq 2n \log M.$$

As defined above, elements of the set $\bar{P}$ are given by the relations

$$\bar{p}_{2i} = [p_i, (K+2)^{i-1}], \qquad \bar{p}_{2i-1} = [p_i, -(K+2)^{i-1}] \qquad (i \in \mathbb{N}_n),$$

where $K = 2(M-1)^2$. Therefore, the time complexity of the algorithm that assigns to the problem 2PC an appropriate statement of the problem 3PC is determined by the complexity

[4]Here and below, the notation $\log x$ is used for the binary logarithm of the number x.

of calculating the powers $(K+2)^{i-1}$, $i \in \mathbb{N}_n$. As is known [12], the time complexity of the multiplication of two positive integers does not exceed $O(N \log N \log\log N)$, where N is the length of the largest factor; in our case,

$$N \le n\log(K+2) \le n\,(\log K + 1) = n\,(\log 2(M-1)^2 + 1) = n\,(2\log(M-1) + 2).$$

Hence, the total time complexity is upper bounded by a polynomial of n and $\log M$, i.e., of Len_1.

The lemma is proved.

The following theorem summarizes the above discussion.

Theorem 3. *The problem* 3PC *is NP-complete in the strong sense.*

Proof. As noted above, any arbitrary covering of the set P by straight lines generates an equinumerous covering of the set $\bar{P}$ by planes. Conversely, by Lemma 3, to any arbitrary covering of the set $\bar{P}$, we can assign a covering of P with cardinality not exceeding the original cardinality. Moreover, it follows from Lemma 4 that the suggested reduction is polynomial. As a result, since the problem 2PC is NP-complete in the strong sense, a similar result is valid for the problem 3PC.

2.2. Case of arbitrary dimension. Let us now consider the problem of covering by hyperplanes in spaces of dimension more than three and show that the problem $(k-1)$PC can be reduced to the problem kPC in polynomial time. Let an instance of the problem $(k-1)$PC be given by a set $P = \{p_1, \dots, p_n\} \subset \mathbb{N}_M^{k-1}$ and a positive integer B. Below, we use the natural isomorphic embedding of the $(k-1)$-dimensional space into the k-dimensional one: $x \in \mathbb{R}^{k-1} \mapsto [x, 0] \in \mathbb{R}^k$; we identify the subsets $P \subset \mathbb{R}^{k-1}$ and $\{[p_1, 0], \dots, [p_n, 0]\} \subset \mathbb{R}^k$. To any point $p_i \in P$, we assign a pair of points in the space $\mathbb{Z}^k$ according to the rule

$$\bar{p}_{2i-1} = [p_i, -w_i], \qquad \bar{p}_{2i} = [p_i, w_i], \tag{2.7}$$

where

$$w_i = (K+2)^{i-1} \qquad \text{and} \qquad K = \left\lceil (k-1)^{\frac{k-1}{2}} (M-1)^{k-1} \right\rceil. \tag{2.8}$$

Thus, we construct the set $\bar{P} \subseteq \mathbb{Z}^k$, which specifies, together with the number B, an instance of the problem kPC. As usual, to substantiate the polynomial reducibility, we show that this construction can be performed in polynomial time and the special problems described above have identical answers.

Evidently, any covering of the set P by hyperplanes in the space $\mathbb{R}^{k-1}$ generates an equinumerous covering of the set $\bar{P} \subset \mathbb{R}^k$. The inverse correspondence requires a substantiation. Let us introduce the following additional notation. Denote by π_0 the hyperplane

$$\{[x, 0] \colon x \in \mathbb{R}^{k-1}\},$$

whose role in the reasoning below is similar to the role of the plane xOy in Section 2.1. For an arbitrary subset $Q \subset \mathbb{R}^k$, we denote by $\mathrm{Pr}_{\pi_0} Q$ the orthogonal projection of the subset Q onto the hyperplane π_0.

Lemma 5. *Let subsets $Q \subset P$ and $\bar{Q} \subset \bar{P}$ satisfy the relation $Q = \mathrm{Pr}_{\pi_0} \bar{Q}$, and let the conditions*

$$|\bar{Q}| \ge k+1, \tag{2.9}$$

$$\dim \mathrm{aff}(\bar{Q}) \le k-1$$

hold. Then, $\dim \operatorname{aff}(Q) \le k-2$.

Proof. Evidently, it is sufficient to consider the case when inequality (2.9) turns into an equality. Let π be a hyperplane from the space $\mathbb{R}^k$ containing the affine hull of the set $\bar{Q}$. By analogy with the proof of Lemma 2, we exclude from consideration the trivial case $|Q| < |\bar{Q}|$, when the set $\bar{Q}$ contains points generated by the same element of the set P. Indeed, let $\bar{Q}' = \{\bar{p}_{2i-1}, \bar{p}_{2i}, \bar{p}_{2j}\} \subset \bar{Q}$ for some $\{i,j\} \subset \mathbb{N}$. By formula (2.7), elements of the subset $\bar{Q}'$ are generated by the points $p_i, p_j \in P$. The choice of the point p_{2j} with an even number as the third element of $\bar{Q}'$ is not a matter of principle, since the case p_{2j-1} can be considered by analogy. Then, however,

$$\{[p_i, 0], [p_j, 0]\} \subset \operatorname{aff}(\bar{Q}') \subset \pi.$$

Since the choice of the point $p_{2j} \in \bar{Q}$ was arbitrary, we conclude that $Q = \operatorname{Pr}_{\pi_0} \bar{Q} \subset \pi$ and, consequently, in view of the condition $\pi_0 \ne \pi$,

$$\dim \operatorname{aff}(Q) \le \dim(\pi_0 \cap \pi) = k-2.$$

Thus, we assume further that $|Q| = |\bar{Q}|$. In addition, without loss of generality, we assume that

$$\bar{Q} = \{\bar{p}_{j_1}, \dots, \bar{p}_{j_{k+1}}\},$$

where

$$\bar{p}_{j_t} = [p_{i_t}, z_{i_t}], \qquad |z_{i_t}| = w_{i_t} \qquad (t \in \mathbb{N}_{k+1}),$$

for some

$$1 \le i_1 < i_2 < \ldots < i_{k+1} \le n,$$

and, consequently, the inequalities

$$|z_{i_t}| \le (K+2)\,|z_{i_{t+1}}| \qquad (t \in \mathbb{N}_k) \tag{2.10}$$

are valid. By the assumption of the lemma, $\dim \operatorname{aff}(\bar{Q}) \le k-1$. Hence, the vectors

$$\bar{p}_{j_2} - \bar{p}_{j_1} = [p_{i_2} - p_{i_1}, z_{i_2} - z_{i_1}], \dots, \bar{p}_{j_{k+1}} - \bar{p}_{j_1} = [p_{i_{k+1}} - p_{i_1}, z_{i_{k+1}} - z_{i_1}]$$

are linearly dependent, and the determinant composed of their coordinates is equal to zero. Let us present this determinant in a convenient form:

$$\Delta = \begin{vmatrix} z_{i_2} - z_{i_1} & z_{i_3} - z_{i_1} & \dots & z_{i_{k+1}} - z_{i_1} \\ p_{i_2} - p_{i_1} & p_{i_3} - p_{i_1} & \dots & p_{i_{k+1}} - p_{i_1} \end{vmatrix};$$

then, we expand it along the first row:

$$\Delta = (z_{i_2} - z_{i_1})\Delta_{i_2} + (-1)^1 (z_{i_3} - z_{i_1})\Delta_{i_3} + \ldots + (-1)^{k-1} (z_{i_{k+1}} - z_{i_1})\Delta_{i_{k+1}}$$

$$= (-1)^{k-1} z_{i_{k+1}} \Delta_{i_{k+1}} + \ldots + (-1)^0 z_{i_2} \Delta_{i_2} - z_{i_1}\big(\Delta_{i_2} + (-1)^1 \Delta_{i_3} + \ldots + (-1)^{k-1}\Delta_{i_{k+1}}\big).$$

To complete the proof of the lemma, it is sufficient to show that $\Delta_{i_t} = 0$ for all $t = 2, \dots, k+1$. Assume, by contradiction, that this is not so and i_t is the greatest number of a nonzero determinant. Let us verify that, in this case, $\Delta \ne 0$. As earlier, the reasoning is essentially based on the fact that the coordinates of the points p_{i_t} are integer and, consequently, all the determinants under consideration are integer as well. In particular, the following condition is valid:

$$(\Delta_{i_t} = 0) \vee \left(1 \le |\Delta_{i_t}| \le K = \left\lceil (k-1)^{\frac{k-1}{2}} (M-1)^{k-1} \right\rceil\right),$$

where the upper estimate follows from the Hadamard inequality. Let us estimate the absolute value of the determinant Δ from below:

$$\begin{aligned}|\Delta| &= |(-1)^{t-2} z_{i_t} \Delta_{i_t} + \ldots + z_{i_2}\Delta_{i_2} - z_{i_1}(\Delta_{i_2} + \ldots + (-1)^{t-2}\Delta_{i_t})| \\ &\ge |z_{i_t}|\,|\Delta_{i_t}| - |z_{i_{t-1}}|\,|\Delta_{i_{t-1}}| - \ldots - |z_{i_2}|\,|\Delta_{i_2}| - |z_{i_1}|\,(|\Delta_{i_2}| + \ldots + |\Delta_{i_t}|) \\ &\ge |z_{i_t}| - K\,|z_{i_{t-1}}| - \ldots - K\,|z_{i_2}| - (t-1)\,K\,|z_{i_1}| = E(t).\end{aligned}$$

Let us show that $E(t) > 0$. The proof is performed by induction on t. The base is $t = 2$. By virtue of (2.10),

$$E(2) = |z_{i_2}| - K\,|z_{i_1}| \ge (K+2)\,|z_{i_1}| - K\,|z_{i_1}| = 2\,|z_{i_1}| > 0.$$

Let the statement be valid for all $s \le t$. We give the proof for $s = t+1$. We obtain

$$\begin{aligned}E(t+1) &= |z_{i_{t+1}}| - K\,|z_{i_t}| - \ldots - K\,|z_{i_2}| - tK\,|z_{i_1}| \\ &\ge (K+2)\,|z_{i_t}| - K\,|z_{i_t}| - \ldots - K\,|z_{i_2}| - tK\,|z_{i_1}| = 2\,|z_{i_t}| - K\,|z_{i_{t-1}}| - \ldots - K\,|z_{i_2}| - tK\,|z_{i_1}| \\ &\ge 2\,(|z_{i_t}| - K\,|z_{i_{t-1}}| - \ldots - K\,|z_{i_2}| - (t-1)K\,|z_{i_1}|) = 2E(t) > 0\end{aligned}$$

by the induction assumption. Thus, it is shown that the inequation $\Delta_{i_t} \ne 0$ for arbitrary $t = 2, \ldots, k+1$ implies $\Delta \ne 0$, which contradicts the condition.

The lemma is proved.

Lemma 6. *Let $\bar{\Pi} = \{\bar{\pi}_1, \ldots, \bar{\pi}_t\}$ be a covering of the set $\bar{P}$ by hyperplanes. Then, the set P also possesses a covering by hyperplanes with cardinality not exceeding t.*

Proof. Let us divide the covering $\bar{\Pi}$ into two classes:

$$\bar{\Pi}_1 = \{\bar{\pi} \in \bar{\Pi} \colon |\bar{\pi} \cap \bar{P}| \ge k+1\}, \qquad \bar{\Pi}_2 = \bar{\Pi} \setminus \bar{\Pi}_1.$$

By construction, to an arbitrary hyperplane $\bar{\pi}_j \in \bar{\Pi}_1$, one can assign the subset $\bar{P}_j = \bar{\pi}_j \cap \bar{P}$ such that $|\bar{P}_j| \ge k+1$. By Lemma 5, in the space $\mathbb{R}^{k-1}$, there exists a hyperplane π_j containing the set $P_j = \{p \in P \colon \bar{p} \in \bar{P}_j\}$. The manifold π_j can be naturally extended to a hyperplane of the space $\mathbb{R}^k$ containing $\bar{P}_j \cup P_j \cup \bar{P}'_j$, where $\bar{P}'_j$ consists of the points symmetric to the elements of the subset $\bar{P}_j$ with respect to the hyperplane π_0.

Let us introduce the notation

$$P_I = \bigcup_{\bar{\pi}_j \in \bar{\Pi}_1} P_j, \qquad P_{II} = P \setminus P_I.$$

As proved above, the set P_I possesses a covering by hyperplanes that is equinumerous to the covering $\bar{\Pi}_1$, whereas no point $\bar{p} \in \bar{P}$ with preimage $p \in P_{II}$ belongs to some element of $\bar{\Pi}_1$. Denote the subset consisting of the points $\bar{p} \in \bar{P}$ with preimages $p \in P_{II}$ by $\bar{P}_{II}$. This subset is covered by elements of $\bar{\Pi}_2$; consequently,

$$|\bar{\Pi}_2| \ge \left\lceil \frac{|\bar{P}_{II}|}{k} \right\rceil = \left\lceil \frac{2|P_{II}|}{k} \right\rceil.$$

On the other hand, the set P_{II}, evidently, possesses a covering by hyperplanes in the space $\mathbb{R}^{k-1}$ with cardinality not exceeding

$$\left\lceil \frac{|P_{II}|}{k-1} \right\rceil \le \left\lceil \frac{2|P_{II}|}{k} \right\rceil,$$

since

$$\frac{|P_{II}|}{k-1} \le \frac{2|P_{II}|}{k}$$

for arbitrary $k \ge 2$ and the function $\lceil \cdot \rceil$ monotonically increases. Thus, it is shown that the set P possesses a covering by hyperplanes with cardinality not exceeding t.

The lemma is proved.

Lemma 7. *The reduction of the problem* $(k-1)$PC *to the problem* kPC *described above can be realized in a time that is a polynomial of the length of the problem* $(k-1)$PC.

Proof. An instance of the problem $(k-1)$PC is given by a set of points

$$P = \{p_1, \ldots, p_n\} \subseteq \mathbb{N}_M^{k-1}$$

and a number $B \in \mathbb{N}$. Consequently, the length of the problem $(k-1)$PC is calculated as

$$\mathrm{Len}_1 = (k-1)\, n \log(M-1) + \log B \ge (k-1)\, n \log(M-1).$$

Elements of the set $\bar{P}$ are given by relations (2.7)–(2.8). The time complexity of the algorithm assigning the corresponding statement of the problem kPC to the problem $(k-1)$PC is determined by the complexity of calculating the powers $(K+2)^{i-1}$, $i \in \mathbb{N}_n$. As noted in Lemma 4, the time complexity of the multiplication of two positive integers does not exceed $O(N \log N \log\log N)$, where N is the length of the largest factor. In the proposed algorithm,

$$N \le n \log(K+2) \le n\,(\log K + 1),$$

where

$$\log K \le \log\bigl(2(k-1)^{\frac{k-1}{2}}(M-1)^{k-1}\bigr) = \frac{k-1}{2}\log(k-1) + (k-1)\log(M-1) + 1.$$

Consequently, the time complexity of the whole algorithm is upper bounded by a polynomial of n and $\log M$, i.e., of Len_1.

The lemma is proved.

Theorem 4. *The problem* kPC *for arbitrary fixed* $k > 2$ *is NP-complete in the strong sense.*

For $k = 3$, the assertion of the theorem coincides with the assertion of Theorem 3 proved in Section 2.1. For $k > 3$, the proof can be obtained by successive application of Lemmas 6 and 7.

3. APPROXIMABILITY

In this section, we discuss an optimization version of the problem kPC for arbitrary $k > 2$ (we call it Min-kPC) and prove that it is Max-SNP-hard. For this, we show that the algorithm (proposed in the preceding section) reducing the problem $(k-1)$PC to the problem kPC (for arbitrary $k > 2$) is an L-reduction. Since, as noted in the introduction, the problem Min-2PC is Max-SNP-hard; this will imply that the problem Min-kPC remains Max-SNP-hard for arbitrary $k > 2$. According to [10], the belonging to the class of Max-SNP-hard problems implies the impossibility of constructing a PTAS for the problem Min-kPC ($k > 2$) under the assumption $P \neq NP$.

Theorem 5. *The proposed reduction of the problem* Min-$(k-1)$PC *to the problem* Min-kPC *for arbitrary $k > 2$ is an L-reduction.*

Proof. According to Definition 2, it is necessary to construct two functions R and S computable by LSPACE-algorithms and specify positive constants α and β such that both properties from the definition of L-reduction hold.

To an instance of the problem Min-$(k-1)$PC, the function R assigns an appropriate instance of the problem Min-kPC by the rule described in the preceding section. The spatial complexity of calculations is defined by the memory size necessary to calculate the last coordinates of points from the set $\bar{P}$. These coordinates are obtained by raising the positive integer $\delta = K + 2$ to the power $i-1$ for $i \in \mathbb{N}_n$. It is known that the multiplication of two positive integers can be performed by an LSPACE-algorithm; namely, if N is the sum of the lengths of the factors, then, for their multiplication, the memory size $O(\log N)$ is sufficient. In our case,

$$N \leq \log\delta + \log\delta^{n-2} < (n-1)(\log K + 1) < (n-1)\Big(\frac{k-1}{2}\log(k-1) + (k-1)\log(M-1) + 2\Big).$$

Since, according to Lemma 7, the length of the problem Min-$(k-1)$PC satisfies the inequality

$$\mathrm{Len}_1 < (k-1)\, n \log(M-1),$$

the function R can be calculated with the use of memory $O(\log \mathrm{Len}_1)$; i.e., it is computable by an LSPACE-algorithm. Moreover, evidently, it is sufficient to take α equal to one.

To an arbitrary admissible solution of the problem Min-kPC, the function S assigns an admissible solution of the problem Min-$(k-1)$PC. The input for S is an admissible solution of the problem Min-kPC specified in the form of a row in which some decomposition $\bar{J} = \bar{J}_1, \dots, \bar{J}_L$ of indices of points of the set $\bar{P}$ is written. Each element of the decomposition is given by a list of indices; the indices are separated from each other by one empty symbol, whereas elements of the decomposition are separated from each other by two empty symbols. We use Lemma 6 to calculate the function S in several steps.

I. The first move along the input tape (from its beginning):

1. Consider some element $\bar{J}_t$ of the decomposition and count the number of indices in it. Since the total number of points in the set $\bar{P}$ equals $2n$, we need $O(\log n)$ of memory to calculate $|\bar{J}_t|$.

1.1. If $|\bar{J}_t| \geq k + 1$, then we successively examine the indices from $\bar{J}_t$:

(a) If the index is even ($2i$), then we write -1 on an auxiliary tape.

(b) If the index is odd ($2i - 1$), then we write 1 on the auxiliary tape.

Then, we move along the input tape from the beginning to the current index and compare the indices written on the input tape with the current one. If we meet an index different from the current index by the value written on the auxiliary tape (-1 or 1 depending on the parity of the current index), then we write nothing on the output tape and pass to the next index on the input tape from the set $\bar{J}_t$. Otherwise, we write on the output tape to J_t the index i by dividing the current index in two and adding the last bit of its binary record to the result.
Note that one element of the decomposition $\bar{J}_t$ of indices of points of the set $\bar{P}$ generates one element of the decomposition J_t of indices of points of the set P.

1.2. If $|\bar{J}_t| < k + 1$, then we skip this element.

As a result, when the input tape is examined to the end, we will have on the output tape the elements $J_1, \dots, J_{L_1}$ of the decomposition under construction for indices of points from P; these elements are constructed by the elements $\bar{J}_1, \dots, \bar{J}_L$ of the original decomposition of indices of points from $\bar{P}$ such that $|\bar{J}_t| \geq k+1$.

II. The second move along the input tape (from its beginning):

1. We organize on the auxiliary tape a counter c, which will count the number of indices written on the output tape during the rescanning, and set it to zero (the necessary memory for such a counter is $O(\log n)$).

2. We consider some element $\bar{J}_t$ of the decomposition and compute $|\bar{J}_t|$.

2.1. If $|\bar{J}_t| \geq k+1$, then we skip it.

2.2. If $|\bar{J}_t| < k+1$, then we successively examine the indices from $\bar{J}_t$:

(a) If the index is even $(2i)$, then we write -1 on the auxiliary tape.

(b) If the index is odd $(2i-1)$, then we write 1 on the auxiliary tape.

Then, we verify whether the index different from the current one by the value written on the auxiliary tape was examined earlier on the input tape. If not, we write i on the output tape. Moreover, if $c < k-1$, then we relate the index i to the current element (under construction) of the decomposition of the set P and increase c by one. If $c = k-1$, then the index i is written to a new element of the decomposition; c is assumed to be equal to one.
As earlier, to perform the described actions on the auxiliary tape, memory $O(\log n)$ is required.

As a result, we obtain a decomposition of indices of points from P on the output tape; moreover, as shown, the function S can be computed by an LSPACE-algorithm. It is sufficient to take $\beta = 1$.

The theorem is proved.

Since the problem Min-2PC is Max-SNP-hard, the problem Min-kPC for arbitrary $k > 2$ is also Max-SNP-hard by Theorem 5. The above reasoning proves the next theorem.

Theorem 6. *The problem* Min-kPC *for arbitrary* $k > 2$ *is* Max-SNP-*hard.*

4. CONCLUSIONS

This paper contains the following main results.

1. It is shown that the problem of covering a finite subset of a k-dimensional numerical space by hyperplanes (kPC) is NP-complete in the strong sense for arbitrary $k > 1$; consequently, its optimization version is Min-kPC NP-hard.

2. The problem Min-kPC is Max-SNP-hard for arbitrary fixed $k > 1$, and, consequently, no PTAS can be developed for this problem if $P \neq NP$.

The following questions remain open.

1. The proof of Theorem 6 is obtained by the substantiation of the L-reduction of the problem Min-$(k-1)$PC to the problem Min-kPC. Consequently, the belonging of the problem Min-k_0PC (for some fixed k_0) to the class Max-SNP implies the belonging of the problem Min-kPC to this class for arbitrary $k > k_0$. In this case, all the listed problems obtain the status of Max-SNP-complete problems. In this connection, it is important to substantiate the L-reduction of some known Max-SNP-complete problem to the problem Min-2PC.

2. The proved impossibility of constructing a PTAS for the problem Min-kPC confirms the significance of developing polynomial (pseudopolynomial) approximation algorithms with guaranteed accuracy estimates for the problem in question.

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